Noise averaging and measurement resolution (or "*A little noise is a good thing*")

James Potzick[a)]
National Institute of Standards and Technology, Gaithersburg, Maryland 20899

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When a continuous quantity is measured with a digital instrument or digitized for further processing, a measurement uncertainty component is incurred from quantization of the continuous variable. This uncertainty can be reduced by oversampling and averaging multiple measurements, but only if there is some noise on the measurand. In this article the optimum noise level is determined, and the subsequent improvement in measurement uncertainty calculated. [S0034-6748(99)04604-3]

INTRODUCTION

When making high accuracy dimensional measurements, great pains are often taken to reduce the vibration level in the measurement apparatus because vibration contributes to the variance of the measurement readings. On the other hand, it is common practice in the audio recording industry to add dither, in the form of noise, to an audio signal prior to analog-to-digital (A/D) conversion to reduce the audible effects of quantization. This suggests that, when A/D resolution is a significant component of measurement uncertainty and the sample rate is high enough to allow oversampling and averaging, heroic attempts at noise reduction may be counterproductive.

ANALOG-TO-DIGITAL CONVERSION

Many measurements involve A/D conversion because the data are often processed in a computer. Voltage A/D converters and displacement measuring interferometers convert continuous quantities voltage or position into digital numbers, with a resolution of one least significant bit (LSB).

The most benign model of an A/D converter produces a discrete output y equal to the continuous input x rounded to the nearest LSB. The transfer function y and quantization error ϵ are

$$y=\mathrm{round}(x), \quad \epsilon=x-y.$$

That is, $y=0$ for any $-\frac{1}{2}\leq x<+\frac{1}{2}$ and $y=1$ for any $\frac{1}{2}\leq x<1\frac{1}{2}$, etc. This transfer function and error function are plotted in Fig. 1. This quantization process produces an average error or bias of zero, but adds measurement uncertainty because the actual error is not known.[1]

MEASUREMENT UNCERTAINTY

In general, the error ϵ is the difference between the true value of the measurand and the indicated value.[2] In the absence of bias the International Organization for Standardization (ISO) expanded ($k=2$) measurement uncertainty is twice the square root of the variance of the error,[3] and can be expressed as

$$u=2\sqrt{\int_{-\infty}^{\infty}p(\epsilon)\epsilon^2 d\epsilon},$$

where $p(\epsilon)$ is the normalized probability density function of the possible values of the error, and the average error is zero. This uncertainty component must eventually be combined with all the others in the measurement process.

From Fig. 1, the error is equally likely to be anywhere between $-\frac{1}{2}$ and $+\frac{1}{2}$ LSB (rectangular probability distribution) so

$$p(\epsilon)=1 \quad \text{for} \quad -\tfrac{1}{2}\leq\epsilon<+\tfrac{1}{2}, \quad \text{and} \quad 0 \quad \text{elsewhere},$$

for any value of x. Then the measurement uncertainty from quantization is

$$u=2\sqrt{\int_{-1/2}^{1/2}\epsilon^2 d\epsilon}=0.577 \quad \text{LSBs}.$$

EFFECTS OF NOISE

Noise can be represented by a unity amplitude random variable *noise* ranging from $-\frac{1}{2}$ to $+\frac{1}{2}$ LSB multiplied by an amplitude A, so the noise amplitude is A LSBs peak to peak. Adding noise to the input x changes the output to

$$y=\mathrm{round}(x+A \quad \mathrm{noise}),$$

while the error is still $\epsilon=x-y$.

If the sampling rate is high enough relative to the rate of change of x, each reading y can be replaced by the average of n readings

$$y_{\mathrm{avg}}=\frac{1}{n}\sum_{i=1}^{n} y_i.$$

The error is now

$$\epsilon_{\mathrm{avg}}=x-y_{\mathrm{avg}}.$$

Figure 2 shows the simulated transfer function with added noise for the case of no averaging (left side) and averaging over 10 readings (right side). The qualitative benefit

[a)]Electronic mail: potzick@nist.gov

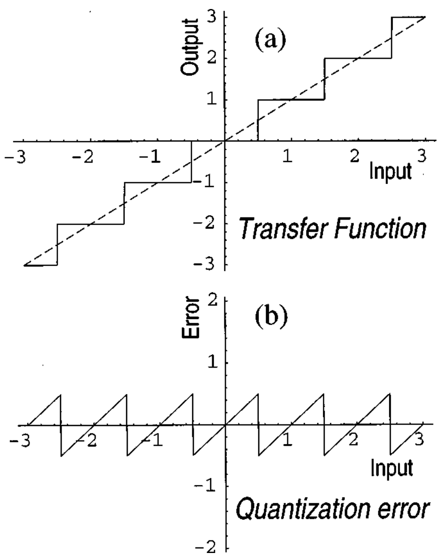

FIG. 1. Transfer function (a) and quantization error (b) for a hypothetical A/D converter. Units are in LSBs.

is immediately apparent. The effort of averaging multiple readings is no added burden because some kind of multiple measurement averaging is always needed to assess the variance of the random error in order to evaluate the measurement uncertainty.

MEASUREMENT UNCERTAINTY WITH NOISE

In the absence of noise the quantization error adds 0.577 LSB of uncertainty to the measurement. The uncertainty can also be found numerically by sampling the random error over a unit interval of x and calculating its statistical variance. This uncertainty is the same for any unit interval of x, and is two times the square root of the variance of the errors

$$u=2\sqrt{\frac{1}{s}\sum_{j=1}^{s}\epsilon_{\text{avg}}^{2}(x_j)}$$

$$\text{for integer} \leqslant x_j < (\text{integer}+1) \quad \text{in } s \text{ steps.}$$

Here ϵ is considered a function of x as x varies over the unit interval. Figure 3 shows the simulated measurement uncer-

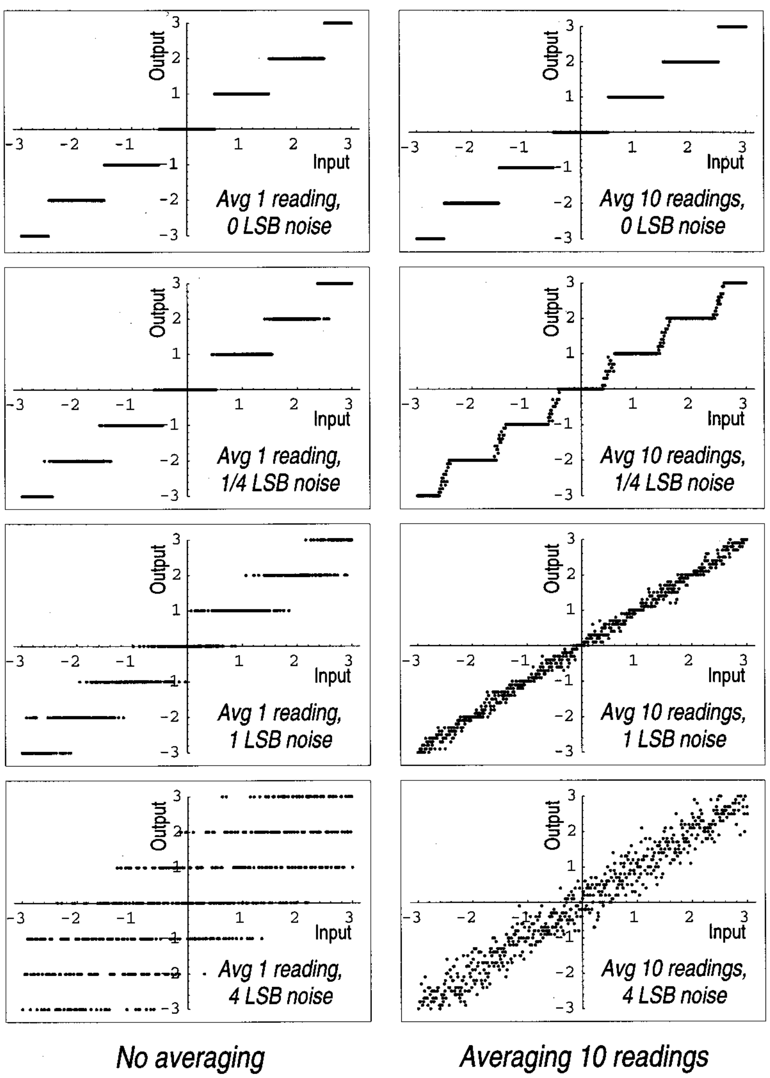

FIG. 2. A/D output with 0, $\frac{1}{4}$, 1, and 4 LSB of noise added to the input, with no averaging (left side) and with each data point the average of 10 readings (right side). Scales are in LSBs.

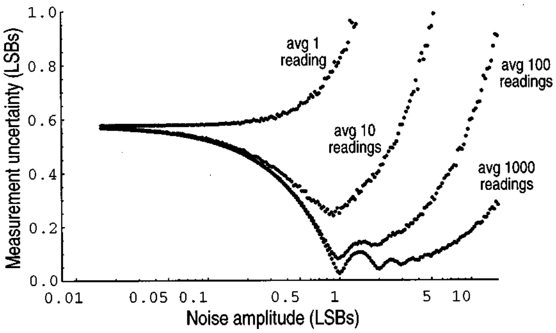

FIG. 3. Combined uncertainty from quantization and noise after averaging over 1, 10, 100, and 1000 readings.

tainty in LSBs for noise amplitude ranging from 0 to 16 LSBs, and averaging over 1, 10, 100, and 1000 readings.

By allowing 1 LSB of noise on the measurand, the uncertainty due to quantization error and noise combined can be reduced from 0.577 LSB with no noise to 0.25, 0.08, or 0.025 by averaging over 10, 100, or 1000 readings, respectively, a reduction by factors of 0.44, 0.14, or 0.04. The benefit is proportional to the square root of the number of readings averaged, as expected.

DISCUSSION

An example application is a scanning microscope used for traceable dimensional measurements in the range from nanometers to centimeters.[4] The specimen stage moves at a constant velocity while on-axis image data (e.g., intensity in an optical microscope) are acquired. The stage position is monitored with a displacement measuring interferometer, and stage displacement and digitized on-axis image data are input to a computer. These two sets of data are represented by two one-dimensional arrays in the computer, and the averaging can be accomplished by replacing each array element with the average of the *n* elements around it. When this is done, the interferometer resolution is no longer relevant and the measurement resolution is limited only by system noise. The measurement resolution may now be far higher than the interferometer resolution.

This averaging process is a simple moving average digital low pass filter, performed in software, after the data have been acquired. The minima appearing at integer values of noise amplitude A in Fig. 3 reflect the $[(\sin x)/x]^2$ amplitude frequency response of the unweighted moving average digital filter. The cutoff frequency is chosen by adjusting the ratio of n to the sampling interval. The contributions of each array element to the average can be weighted and normalized to alter the frequency response of the filter (in x space). The cutoff frequency should be chosen to pass the highest meaningful frequencies in the data and reject all frequencies above that. In this case the highest spatial frequency is fixed by the microscope resolution, which is limited by the objective numeric aperture and illumination wavelength. The low pass filter can be chosen to remove all frequencies above that, including that part of the added noise. The noise can be random or periodic, but it must not be correlated with the sampling interval.

In this example, measurement accuracy was improved without incurring the cost of a higher resolution interferometer or imposing extensive vibration reduction measures.

The quantization error can be reduced by allowing (or adding, if necessary) 1 LSB of random noise to the measurand and averaging oversampled readings.

The benefits of noise averaging are reduced measurement uncertainty and increased resolution. The costs are the requirement for a higher sampling rate for a given data frequency limit (or longer acquisition time and lower data frequency response) and some digital processing. The noise is free.

[1] I. H. Lira and W. Wöger, Meas. Sci. Technol. **8**, 441 (1997).
[2] *International Vocabulary of Basic and General Terms in Metrology*, 2nd ed. (International Organization for Standardization, Geneva, 1993).
[3] *Guide to the Expression of Uncertainty in Measurement*, 1st ed. (International Organization for Standardization, Geneva, 1993).
[4] J. Potzick, Antireflecting-Chromium Linewidth Standard, SRM 473, for Calibration of Optical Microscope Linewidth Measuring Systems, NIST Special Publication No. SP-260-129 (1997).