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XCVII. *The Demagnetizing Factors for Ellipsoids.*

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1. *Introductory.*

Formulæ for the demagnetizing field in a uniformly magnetized ellipsoid of revolution, first derived by Maxwell, are given in many books, but the numerical evaluation of the expressions, although it presents no essential difficulty, is somewhat troublesome. No comprehensive table of values seems to have been published, and the short lists of illustrative values which are sometimes given are seldom free from significant errors. As the need for a reliable set of values of demagnetizing coefficients for ellipsoids makes itself felt in connection with a variety of magnetic problems, it was thought that a useful purpose might be served by carrying out the calculations systematically over the whole of the range of the parameters with a degree of accuracy which would ensure adequacy for any likely theoretical or practical application. The results of the calculations are embodied in the tables in this paper.

The direct practical application of the demagnetizing factors for ellipsoids, except for such special cases as the sphere and the disc, is rather limited, though these factors are required in connection with a number of methods which have been used for measurements on ferromagnetics. In spite of the difficulty of making accurately ellipsoidal test pieces, the advantages, for precision work, of uniformity of magnetization throughout the specimen, would seem to make the development of methods employing them worthy of fuller consideration. Indirectly, the exactly calculable fields in magnetized ellipsoids are of very considerable value in providing a guide in estimating the field and intensity distribution in the variously shaped pieces of magnetic material used in magnetic and electromagnetic equipment. Moreover, the range of shapes covered by the term ellipsoid of revolution is sufficiently wide to include approximations to most of the shapes of portions of ferromagnetic material which it may be desirable to consider in connection with the internal structure of ferromagnetic metals and alloys.

The form in which the general expressions for the demagnetizing factors of the ellipsoid with three unequal axes is given by Maxwell and later writers is not appropriate for direct numerical evaluation. These demagnetizing factors seldom come into use, and the labour of drawing up

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extensive numerical tables would be out of all proportion to their usefulness. To supplement the numerical tables for ellipsoids of revolution, however, it has seemed worth while to re-express the Maxwell formulæ in terms of elliptic integrals of standard form, so that with the aid of readily available tables, numerical values can be obtained, when required, with a minimum of difficulty. Moreover the interesting inter-relations between the general demagnetizing factors, for three unequal axes, and the special factors, for two axes equal, are not clearly or fully brought out by the expressions which have previously been given.

For a body which is uniformly magnetized, with intensity of magnetization I, the usual demagnetizing coefficient, N, may be defined by the relation

$$H_t = H_a - NI, \quad \ldots \quad (1.1)$$

where H_a is the applied field, and H_t the total field in the material (the "pipe" field), H_t, H_a, and I being co-directional. Except in the special case where the specimen is magnetized to saturation in very strong fields, uniformity of magnetization is possible in a uniform applied field only for an ellipsoid. (Approximate uniformity of magnetization may be obtained by the use of non-uniform fields along limited lengths of bars as in some permeameter arrangements.) The field, H, due to the magnetization alone (putting H_a in (1.1) equal to zero) is co-directional with I for magnetization along the principal axes; taking these as a, b, c along the x, y, z directions, the components of the field due to magnetization of intensity I are given by

$$H_x = -N_a I_x, \quad H_y = -N_b I_y, \quad H_z = -N_c I_z, \quad \ldots \quad (1.2)$$

N_a, N_b and N_c being the demagnetizing coefficients. Except for the sphere, the resultant field, H, will not in general be co-directional with the intensity I. It may be noted that

$$N_a + N_b + N_c = 4\pi. \quad \ldots \quad (1.3)$$

Although the coefficients, N, defined as above, have usually been used in the past, there are several advantages in using a modified factor, D, defined by

$$D = N/4\pi. \quad \ldots \quad (1.4)$$

The numerical relations between the various factors then become more immediately obvious, as illustrated by the form taken by (1.3), namely,

$$D_a + D_b + D_c = 1. \quad \ldots \quad (1.5)$$

Moreover, from a practical standpoint, the use of D is no less convenient than N. The field due to the magnetization is given, in terms of D, by

$$H = -D(4\pi I) = -D(B-H) = DB/(1-D) \quad \ldots \quad (1.6)$$

With an applied field, H_a, the field, H_t, in the specimen (see (1.1)) is given by

$$H_t = H_a - \frac{D(B-H_a)}{1-D} \quad \ldots \quad (1.7)$$

When, as is most usual, data are given for B rather than I, division by factors such as $4\pi - N$ is necessary if the coefficients N are used; little purpose is therefore served by multiplying the factors D, which are first evaluated, by 4π, when this, more often than not, merely renders the calculations made in applications more troublesome. In the sequel, therefore, the factors D will be used exclusively, and referred to as demagnetizing factors.

In a field which has been so long and so intensively explored as that involving the geometry of ellipsoids, fundamentally new results are not to be expected. It so happens, however, that although the physical problem of the uniformly magnetized ellipsoid was solved, in a fundamental sense, some seventy years ago, the results have remained, to a large extent, in a form which is neither readily intelligible nor readily applicable. The present paper is a contribution to the filling of a gap between fundamental theory and its practical application.

2. *Outline of Derivation of General Formulæ.*

The problem of magnetic induction in an ellipsoid is presented by Maxwell ('Treatise,' Articles 437 and 438) in a very simple manner and his treatment will be outlined in this paragraph. Following a method due to Poisson, the theoretical determination of the magnetic potential in a uniformly magnetized body is shown to be mathematically equivalent to the determination of the gravitational field in a body of the same shape and of uniform density. In Maxwell's words, "If V is the potential at a point (x, y, z), due to the gravitation of a body of any form of uniform density ρ, then $-dV/dx$ is the potential of the same body if uniformly magnetized in the direction of x with the intensity $I = \rho'$." The proof given is to the effect that $-(dV/dx)\delta x$ would be the potential due to a body of density ρ together with a body of the same shape and density $-\rho$ shifted a distance $-\delta x$ relative to the first; elements $\rho\delta v$ and $-\rho\delta v$ with the second element shifted $-\delta x$ relative to the first, are equivalent to a magnetic element of moment $\rho\delta v\delta x$, the intensity of magnetization of the element being $\rho\delta x$; hence $-(dV/dx)\delta x$ is the potential due to a body magnetized with the intensity $\rho\delta x$ in the direction of x, and $-dV/dx$ is that of a body magnetized with intensity ρ. If the magnetization is to be uniform and hence also the field due to it (the applied field is assumed to be uniform, the value zero being tacitly included), the magnetic potential, Ω, must be a linear function of the co-ordinates x, y, z within the body, and the corresponding "gravitational" potential V must be a quadratic function of the co-ordinates. To quote Maxwell again, "The only cases with which we are acquainted in which V is a quadratic function of the co-ordinates within the body are those in which the body is bounded by a complete surface of the second degree, and the only case in which such a body is of finite dimensions is when it is an ellipsoid." The standard integral expression, in a form equivalent to that given below, for the potential of an ellipsoid is then introduced, with a reference to Thomson and Tait's 'Natural Philosophy,' and the integral forms of the expressions for the demagnetizing coefficients follow immediately. The explicit expressions which may be obtained for these coefficients for ellipsoids of revolution are then given in the often quoted forms, involving the eccentricities.

The inter-relation between the magnetic potential of a uniformly magnetized body and the gravitational potential of a body of the same shape of uniform density may be shown generally and with brevity by using vector notation (*cf.* Stoner, 'Magnetism and Matter,' p. 28, 1934).

The symbol ∇ will be used, where

$$\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z},$$

i, **j** and **k** being unit vectors in the x, y, z directions. The dot symbol is used to denote a scalar product. It may be noted that $\nabla\cdot$ applied to a vector is equivalent to div, and ∇ applied to a scalar to grad. A general expression for the magnetic potential, Ω, due to a magnetized body (obtained by simple generalization of the expression for the potential due to a dipole) is

$$\Omega=-\int \mathbf{I}\cdot\nabla\left(\frac{1}{r}\right)d\mathrm{V}, \qquad (2.1)$$

which for uniform magnetization becomes

$$\Omega=-\mathbf{I}\cdot\int\nabla\left(\frac{1}{r}\right)d\mathrm{V}. \qquad (2.2)$$

Since $-\int\nabla\left(\frac{1}{r}\right)d\mathrm{V}$ gives the gravitational force due to a volume of uniform unit density of mass (apart from the constant G), or the electric force due to a volume of uniform unit density of charge, the required theorem is immediately proved. It may be noticed that (2.1) may be transformed by Green's theorem to give the potential in terms of Poisson's equivalent surface and volume distributions. With $\mathbf{n}_1$ as the unit normal to the surface element $d\mathrm{S}$,

$$\Omega=\int\frac{1}{r}(\mathbf{I}\cdot\mathbf{n}_1)\,d\mathrm{S}-\int\frac{1}{r}(\nabla\cdot\mathbf{I})\,d\mathrm{V}. \qquad (2.3)$$

For uniform magnetization, the second term vanishes, and the distribution can be treated as a surface distribution of magnetic charge of pole strength per unit area $\mathbf{I}\cdot\mathbf{n}_1$ (equal to I cos θ, where θ is the angle between I and the outward drawn normal). The great difficulty in predetermining the fields due to magnetized materials in forms of practical interest, such as rods and bars and electromagnet pole pieces, is that the second term is usually comparable with the first, and it is only by exceedingly laborious methods that even rough approximations to the distribution and thence to the value of the integral can be obtained *. The ellipsoid is troublesome to deal with ; but it is easier than anything else.

* A good account of work in the latter part of the last century on the general problem of demagnetizing fields is given in the book by Du Bois (1896), and in a paper by Bozorth and Chapin (1942) on the demagnetizing factors of rods references are given to more recent work.

The potential at an internal point of an ellipsoid of unit uniform density, with semi-axes a, b, c in the x, y, z directions respectively may be expressed in the following form which was first given by Dirichlet in 1839 :

$$\mathrm{V}=\pi abc\int_0^\infty\left\{1-\frac{x^2}{a^2+s}-\frac{y^2}{b^2+s}-\frac{z^2}{c^2+s}\right\}\frac{ds}{(a^2+s)^{1/2}(b^2+s)^{1/2}(c^2+s)^{1/2}}. \quad (2.4)$$

There is no short cut to this formula, and for its derivation reference must be made to treatises dealing with potential theory. A detailed treatment of this classical problem of the ellipsoid is given, for example, in Thomson and Tait's 'Natural Philosophy' (vol. i. part ii., articles 494 and 519–533) and in Webster's 'Dynamics' (articles 151–161), in both of which books an indication is given of the historical sequence of development of the theory. Certain aspects of the mathematical theory are discussed fully in Jeans's 'Electricity and Magnetism' in connection with the problem of an ellipsoid in a uniform electric field, and a more succinct treatment with reference to the present problem is given in Stratton's 'Electromagnetic Theory' (sections 3.27 and 4.18).

From the relation proved above it follows that the x component of the demagnetizing field for a uniformly magnetized ellipsoid is given by

$$\mathrm{H}_x=-\frac{\partial\Omega}{\partial x}=\frac{\partial^2\mathrm{V}}{\partial x^2}=-2\pi abc\mathrm{I}_x\int_0^\infty\frac{ds}{(a^2+s)^{3/2}(b^2+s)^{1/2}(c^2+s)^{1/2}}, \quad (2.5)$$

with similar expressions for H_y and H_z. The demagnetizing factors, as defined in (1.2) and (1.4), are therefore given by

$$\mathrm{D}_\nu=\frac{abc}{2}\int_0^\infty\frac{ds}{(\nu^2+s)\mathrm{R}_s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.6)$$

where $\nu=a,\ b,\ c$ and $\mathrm{R}_s=\{(a^2+s)(b^2+s)(c^2+s)\}^{1/2}$.

These are the general integral expressions for the demagnetizing factors of an ellipsoid.

3. *Expression of Demagnetizing Factors in Terms of Normal Elliptic Integrals.*

The integrals in the formulæ (2.6) for the demagnetizing factors of an ellipsoid are elliptic integrals, and for their evaluation it is appropriate to express them in terms of elliptic integrals which have been tabulated. The tabulated integrals are Legendre's normal integrals of the first and second kinds, namely,

$$\mathrm{F}(k,\phi)=\int_0^\phi\frac{d\psi}{(1-k^2\sin^2\psi)^{1/2}}=\int_0^x\frac{dz}{(1-z^2)^{1/2}(1-k^2z^2)^{1/2}}, \quad . \quad (3.1)$$

and

$$\mathrm{E}(k,\phi)=\int_0^\phi(1-k^2\sin^2\psi)^{1/2}d\psi=\int_0^x\frac{(1-k^2z^2)^{1/2}\,dz}{(1-z^2)^{1/2}}, \quad . \quad . \quad (3.2)$$

where $k^2<1$, $x=\sin\phi$.

Taking $a\geqslant b\geqslant c$, by the substitution $z^2=(a^2-c^2)/(a^2+s)$, the expressions (2.6) become

$$\mathrm{D}_\nu=\frac{abc}{(a^2-c^2)^{1/2}}\int_0^x\left[z^2\Big/\left\{\left(1-\frac{a^2-\nu^2}{a^2-c^2}z^2\right)^{1/2}(1-z^2)^{1/2}(1-k^2z^2)^{1/2}\right\}\right]dz, \quad (3.3)$$

where $k^2=(a^2-b^2)/(a^2-c^2)$, $x^2=1-c^2/a^2$.

For $\nu=a$, the required transformation is immediately effected since

$$\frac{z^2}{(1-z^2)^{1/2}(1-k^2z^2)^{1/2}}=\frac{1}{k^2}\left\{\frac{1}{(1-z^2)^{1/2}(1-k^2z^2)^{1/2}}-\frac{(1-k^2z^2)^{1/2}}{(1-z^2)^{1/2}}\right\}.$$

For $\nu=b$, an integration by parts is necessary, and the required form is most readily obtained by using the relation

$$\frac{d}{dz}\left\{\frac{z(1-z^2)^{1/2}}{(1-k^2z^2)^{1/2}}\right\}=\frac{(1-z^2)^{1/2}}{(1-k^2z^2)^{1/2}}-(1-k^2)\frac{z^2}{(1-z^2)^{1/2}(1-k^2z^2)^{3/2}}.$$

For $\nu=c$, a similar procedure may be followed, using the relation

$$\frac{d}{dz}\left\{\frac{z(1-k^2z^2)^{1/2}}{(1-z^2)^{1/2}}\right\}=\frac{(1-k^2z^2)^{1/2}}{(1-z^2)^{1/2}}+(1-k^2)\frac{z^2}{(1-z^2)^{3/2}(1-k^2z^2)^{1/2}}.$$

The formulæ finally obtained for the demagnetizing factors in terms of F and E are

$$\left.\begin{aligned}
D_a&=\frac{abc}{(a^2-c^2)^{1/2}(a^2-b^2)}\{F(k,\phi)-E(k,\phi)\},\\
D_b&=-\frac{abc}{(a^2-c^2)^{1/2}(a^2-b^2)}\{F(k,\phi)-E(k,\phi)\}\\
&\qquad+\frac{abc}{(a^2-c^2)^{1/2}(b^2-c^2)}E(k,\phi)-\frac{c^2}{b^2-c^2},\\
D_c&=-\frac{abc}{(a^2-c^2)^{1/2}(b^2-c^2)}E(k,\phi)+\frac{b^2}{b^2-c^2},
\end{aligned}\right\}\quad . \quad . \quad (3.4)$$

where $k^2=(a^2-b^2)/(a^2-c^2),\quad \sin^2\phi=x^2=1-c^2/a^2.$

Since the demagnetizing factors depend on the relative and not the absolute lengths of the semi-axes, it is convenient to express them in terms of dimensional ratios. The final formulæ are therefore set out below in this form.

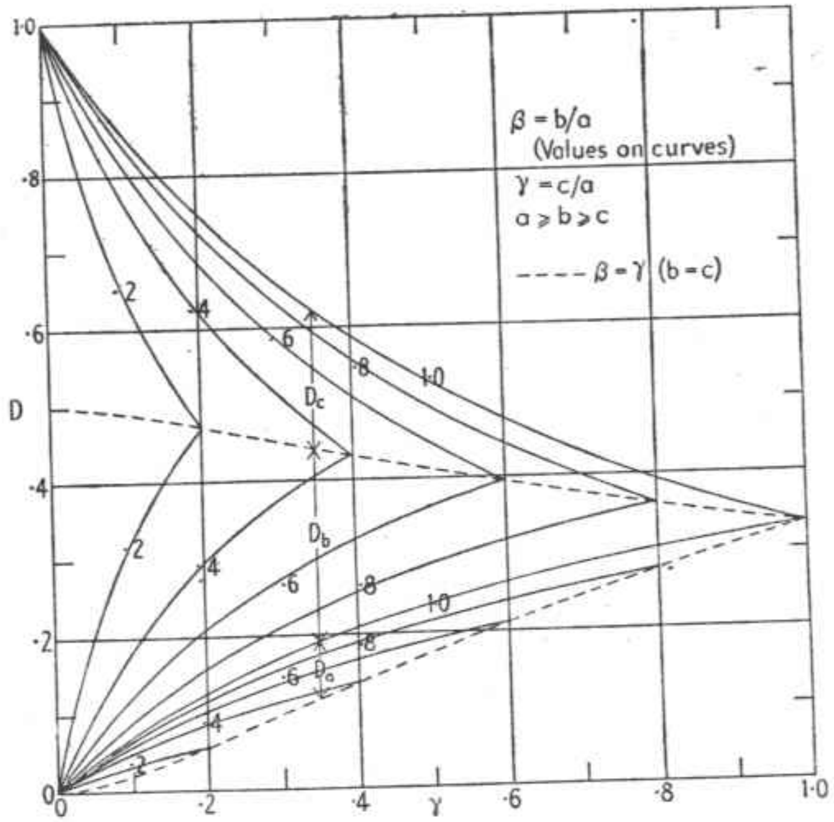
For an ellipsoid of semi-axes a, b, c $(a\geqslant b\geqslant c)$, let $b/a=\beta$, $c/a=\gamma(1\geqslant\beta\geqslant\gamma)$. Then

$$\left.\begin{aligned}
D_a&=\frac{\beta\gamma}{(1-\gamma^2)^{1/2}(1-\beta^2)}\{F(k,\phi)-E(k,\phi)\},\\
D_b&=-\frac{\beta\gamma}{(1-\gamma^2)^{1/2}(1-\beta^2)}\{F(k,\phi)-E(k,\phi)\}\\
&\qquad+\frac{\beta\gamma}{(1-\gamma^2)^{1/2}(\beta^2-\gamma^2)}E(k,\phi)-\frac{\gamma^2}{\beta^2-\gamma^2},\\
D_c&=-\frac{\beta\gamma}{(1-\gamma^2)^{1/2}(\beta^2-\gamma^2)}E(k,\phi)+\frac{\beta^2}{\beta^2-\gamma^2},
\end{aligned}\right\}\quad . \quad . \quad (3.5)$$

where $k^2=(1-\beta^2)/(1-\gamma^2),\quad \sin^2\phi=x^2=1-\gamma^2.$

The arguments in tables of incomplete elliptic integrals $F(k,\phi)$ and $E(k,\phi)$ are usually ϕ and θ where $\theta=\sin^{-1}k$. Suitable tables are given in the collections of Dale (1937), Hayashi (1930) (5-place), and Jahncke-Emde (1933) (4-place). In all of these, values are given for every degree of ϕ, and every fifth degree of θ. The 9-place tables given in Legendre's treatise (1825–8) at intervals of one degree in both arguments have been reprinted (1931). The evaluation of the demagnetizing factors for given values of the dimensional ratios, using (3.5), involves a double interpolation in the tables of integrals, and, with the intervals used in the tables, higher differences must be taken into account if an accuracy corresponding to the tabular accuracy is required in the interpolates. The rapid increase of $F(k, \phi)$ as k and $\sin\phi$ approach unity would make the use of the tables in this region inappropriate, but here series developments could be obtained for each demagnetizing factor, the leading term being that corresponding to the factor for the ellipsoid of revolution to which the general ellipsoid approximates.

Fig. 1.



Demagnetizing factors for ellipsoids.

The demagnetizing field, H, is given by $H=-4\pi DI$. Curves are drawn for varying γ, with β constant. The broken curves do not fall in this series, but are limiting curves for $\beta=\gamma$. With $a\geqslant b\geqslant c$, $D_a\leqslant D_b\leqslant D_c$. The regions in which the values fall are indicated.

Extensive numerical tables of D_a, D_b and D_c are unlikely to be required, but a sufficient number of values have been obtained, to 3- or 4-place accuracy, to make clear the character of the dependence of the values on the dimensional ratios $\beta(=b/a)$ and $\gamma(=c/a)$. The curves obtained are shown in fig. 1.

In fig. 1 the whole range of values is covered, though at the wide interval of 0·2 in β. The two broken curves for $\beta=\gamma$ give the demagnetizing factors for prolate ellipsoids of revolution, the lower curve along the polar axis, the upper along the equatorial axis, as functions of the ratio of the shorter (equatorial) to the longer (polar) axis. The two curves for $\beta=1$ (i. e., $a=b$) give the demagnetizing factors for oblate ellipsoids of revolution, the upper along the polar axis, the lower along the equatorial axis, as functions of the ratio of the shorter (polar) to the longer (equatorial) axis. The curves all meet at the value 1/3, corresponding to a sphere. The limiting values on the left-hand side of the figure for $\gamma\rightarrow 0$ depend on how that limit is approached. For $1>\beta>\gamma$, the limiting values, for $\gamma\rightarrow 0$, are 1, 0 and 0 for D_c, D_b, and D_a respectively; for $\beta=\gamma$, the limiting values are 0·5, 0·5 and 0. For given values of β and γ the sum of the three demagnetizing factors, as shown by (3.4) or (3.5), is unity. This is brought out clearly in the figure, in which, it should be noticed, the lower of the $\beta=1{\cdot}0$ curves consists of coincident curves for D_a and D_b, and the upper broken curve of coincident curves for D_b and D_c.

4. *The Ellipsoid of Revolution.*

Derivation of Formulæ.—For ellipsoids of revolution the elliptic integrals in the general expressions (2.6) for the demagnetizing factors reduce to inverse circular or logarithmic functions. Formulæ may be obtained from (3.4) or (3.5) for the limits $b\rightarrow a$ (oblate ellipsoid) and $c\rightarrow b$ (prolate ellipsoid), the first corresqonding to $k^2\rightarrow 0$, the second to $k'^2=1-k^2\rightarrow 0$, by expanding the intɘgrands of $F(k, \phi)$ and $E(k, \phi)$ as power series in k^2 or k'^2 as appropriate. It is, however, simpler to proceed directly from the basic integrals (2.6), putting $a=b$, or $b=c$, as appropriate, or, for the special case of the sphere $a=b=c$.

To avoid the use of unusual symbols, the convention $a\geqslant b\geqslant c$ used in section 3 will not be retained, and in connection with the ellipsoid of revolution, treated in this section, the following symbols will be used:

a, polar semi-axis; b, equatorial semi-axis; $m=a/b$; $\mu=1/m=b/a$.

The entire range is covered by $0\leqslant m\leqslant\infty$, or by $0\leqslant\mu\leqslant\infty$, or by $0\leqslant m\leqslant 1$ together with $0\leqslant\mu\leqslant 1$.

The general formulae (2.6) become

$$\left.\begin{aligned} D_a&=\frac{ab^2}{2}\int_0^\infty\frac{ds}{(a^2+s)^{3/2}(b^2+s)},\\ D_b&=\frac{ab^2}{2}\int_0^\infty\frac{ds}{(a^2+s)^{1/2}(b^2+s)^2}.\end{aligned}\right\}\quad . \; . \; . \; . \; . \; . \quad (4.1)$$

It is convenient, though not essential, in evaluating the integrals, to treat separately the cases $\mu<1$ and $m<1$.

A. *Prolate spheroid.*

$$a>b,\quad a/b=m>1,\quad b/a=\mu<1.$$

The substitution $z^2=(a^2-b^2)/(a^2+s)$ in (4.1) results in

$$D_a=\frac{\mu^2}{(1-\mu^2)^{3/2}}\int_0^{(1-\mu^2)^{1/2}}\frac{z^2\,dz}{1-z^2},$$

$$D_b=\frac{\mu^2}{(1-\mu^2)^{3/2}}\int_0^{(1-\mu^2)^{1/2}}\frac{z^2\,dz}{(1-z^2)^2}.$$

In the expression for D_a the integral may be rewritten

$$\frac{z^2}{1-z^2}=\frac{1}{2}\left(\frac{1}{1+z}+\frac{1}{1-z}\right)-1$$

and the integration immediately effected. In the expression for D_b, an integration by parts is required, using the relation

$$\frac{d}{dz}\frac{z}{1-z^2}=\frac{1}{1-z^2}+\frac{2z^2}{(1-z^2)^2}.$$

The final formulae may be put in the forms :

$$D_a=\frac{\mu^2}{1-\mu^2}\left[\frac{1}{(1-\mu^2)^{1/2}}\ln\left\{\frac{1+(1-\mu^2)^{1/2}}{\mu}\right\}-1\right],$$

$$=\frac{1}{(m^2-1)}\left[\frac{m}{(m^2-1)^{1/2}}\ln\{m+(m^2-1)^{1/2}\}-1\right]; \quad . \quad . \quad . \quad (4.2)$$

$$D_b=\frac{1}{2(m^2-1)}\left[m^2-\frac{m}{(m^2-1)^{1/2}}\ln\{m+(m^2-1)^{1/2}\}\right],$$

$$=\frac{1}{2}(1-D_a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

The form (4.2) is suitable for numerical evaluation. The simple relation (4.3), a special case of the general relation (1.5), makes separate evaluation (and tabulation) of D_b unnecessary.

For the prolate spheroid the eccentricity, ϵ, is given by

$$\epsilon^2=1-(b/a)^2=1-\mu^2=(m^2-1)/m^2.$$

In terms of the eccentricity, the demagnetizing factor is given by

$$D_a=\frac{1-\epsilon^2}{\epsilon^2}\left\{\frac{1}{2\epsilon}\ln\frac{1+\epsilon}{1-\epsilon}-1\right\}, \quad . \quad . \quad . \quad . \quad . \quad (4.4)$$

the form originally given by Maxwell.

B. *Oblate spheroid.*

$$a<b, \quad a/b=m<1, \quad b/a=\mu>1.$$

The substitution $z^2=(b^2-a^2)/(b^2+s)$ in (4.1) results in

$$D_a=\frac{m}{(1-m^2)^{3/2}}\int_0^{(1-m^2)^{1/2}}\frac{z^2\,dz}{(1-z^2)^{1/2}}.$$

Use of the relation

$$\frac{d}{dz}\frac{z}{(1-z^2)^{1/2}}=\frac{z^2}{(1-z^2)^{3/2}}+\frac{1}{(1-z^2)^{1/2}}$$

reduces the integral to the standard form

$$\int(1-z^2)^{-1/2}\,dz=\sin^{-1}z,$$

leading to the formula

$$D_a=\frac{1}{1-m^2}\left\{1-\frac{m}{(1-m^2)^{1/2}}\cos^{-1}m\right\}, \quad . \quad . \quad . \quad . \quad (4.5)$$

a form suitable for numerical evaluation. D_b is obtained using (4.3).

For the oblate spheroid the eccentricity, ϵ, is given by

$$\epsilon^2=1-(a/b)^2=1-m^2=(\mu^2-1)/\mu^2.$$

In terms of the eccentricity, the demagnetizing factor is given by

$$D_a=\frac{1}{\epsilon^2}\left\{1-\frac{(1-\epsilon^2)^{1/2}}{\epsilon}\sin^{-1}\epsilon\right\}, \quad . \quad . \quad . \quad . \quad . \quad (4.6)$$

the Maxwell form.

Relation between factors for $m>1$ and $m<1$.—The formula (4.2) is in a practically useful form for $m>1$ ($\mu<1$), but its formal validity is not restricted to this range, and it is of interest to obtain, from this single formula, formulae appropriate to the two ranges $m>1$ and $m<1$, and of similar character. From the equations

$$x=\cosh u=\tfrac{1}{2}(e^u+e^{-u})$$

and

$$x=\cos u=\tfrac{1}{2}(e^{iu}+e^{-iu}),$$

it is readily shown that

$$\cosh^{-1}x=\ln\{x+(x^2-1)^{1/2}\}\,;\ \cos^{-1}x=-i\ln\{x+i(1-x^2)^{1/2}\}.$$

Using these relations the formula (4.2) can at once be put in the form

$$D_a=\frac{1}{m^2-1}\left\{\frac{m}{(m^2-1)^{1/2}}\cosh^{-1}m-1\right\}, \quad . \quad . \quad . \quad . \quad (4.7)$$

appropriate for $m>1$, and in the form

$$D_a=\frac{1}{1-m^2}\left\{1-\frac{m}{(1-m^2)^{1/2}}\cos^{-1}m\right\}, \quad . \quad . \quad . \quad . \quad (4.8)$$

appropriate for $m<1$.

C. *Sphere.*—The formula (4.1) becomes

$$D_a=\frac{a^3}{2}\int_0^\infty\frac{ds}{(a^2+s)^{5/2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.9)$$

which integrates immediately to give

$$D_a=\frac{1}{3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.10)$$

a result usually obtained by elementary methods dependent essentially on (2.3).

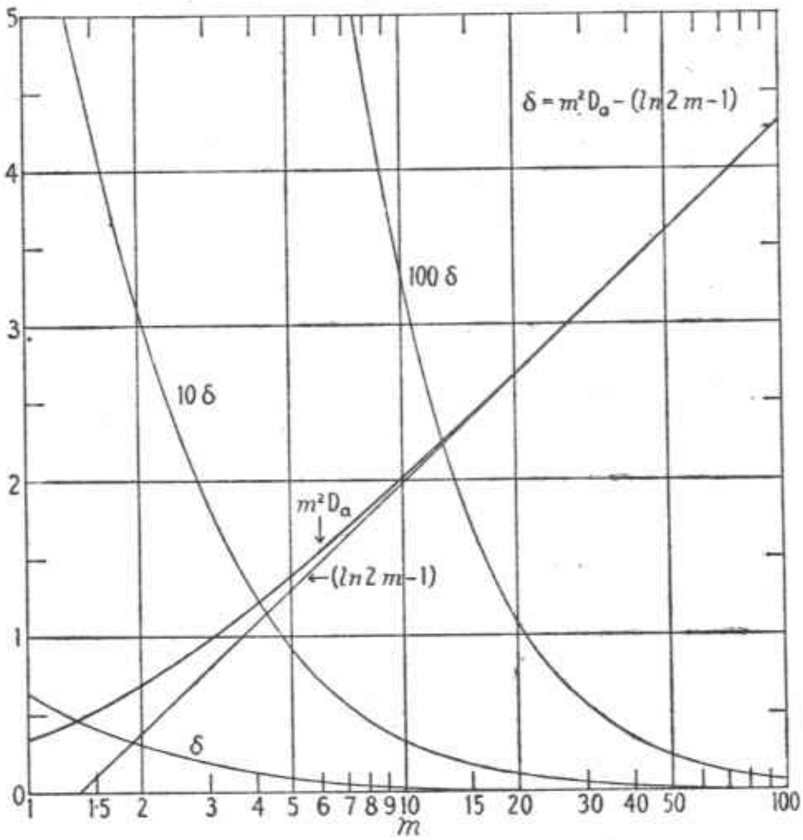
Series Expansions.—Series expansions are of value in showing the limiting forms of the expressions for the demagnetizing factors, in indicating in an obvious way the character of the variation, and, for certain ranges, in facilitating numerical evaluation. They may be developed from the series for the logarithmic and inverse circular functions which are given in many books (*e. g.*, Dwight, 1934; Milne-Thomson and Comrie, 1931). Usually the general expressions for the coefficients are cumbersome, and it is probably more useful to give explicitly the coefficients for the leading terms.

Elongated Prolate Spheroid.—$m \gg 1$, $\mu = 1/m \ll 1$.

$$D_a = \mu^2 \left\{ \left(1 + \frac{3}{2}\mu^2 + \frac{15}{8}\mu^4 + \frac{35}{16}\mu^6 + \dots\right) \ln 2m - \left(1 + \frac{5}{4}\mu^2 + \frac{47}{32}\mu^4 + \frac{319}{192}\mu^6 + \dots\right)\right\}, \quad . \quad . \quad . \quad (4.11\,a)$$

$$= \mu^2(\ln 2m - 1) + \tfrac{1}{4}\mu^4(3\ln 2m - 5) + \frac{1}{32}\mu^6(15 \ln 2m - 47) + \frac{1}{192}\mu^8(420 \ln 2m - 319) + \dots \quad (4.11\,b)$$

Fig. 2.



Demagnetizing factors for prolate spheroids.

$m = a/b$; a, polar semi-axis; b, equatorial semi-axis. D_a, demagnetizing factor for polar axis. m^2D_a is plotted against m (logarithmic). For large values of m, m^2D_a approximates to $\ln 2m - 1$. The differences between m^2D_a and $\ln 2m - 1$ are shown on three scales.

This series up to the term in μ^8 gives D_a to an accuracy of 1 unit in the eighth place at $m = 10(\mu = 0{\cdot}1)$, and to 1 unit in the sixth place for $m = 5(\mu = 0{\cdot}2)$. For larger values of m (or smaller values of μ), for which the series is useful in practice, the accuracy obtainable by breaking off at any term can readily be estimated from the next term.

The first term in the series in the form (4.11 *b*) is well known. The character of the approximation provided by this single term is shown by the logarithmic plot in fig. 2. At $m = 10$, the error is less than 1·7 per cent., at $m = 100$ less than 0·016 per cent.

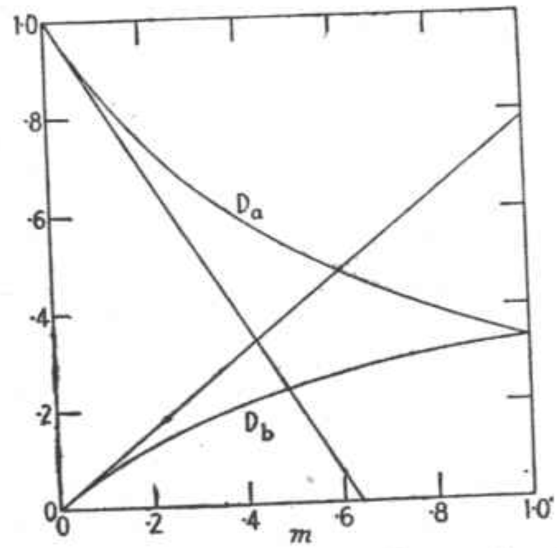
The demagnetizing factor, D_b, along the equatorial axis, cannot be conveniently represented with the form of graph used in fig. 2, but the two broken curves in fig. 1 give D as a function of μ, the lower D_a and the upper D_b, for the range $0 \leqslant \mu \leqslant 1$. A double logarithmic plot, which has been used by Bozorth and Chapin (1942), has advantages for some purposes.

Flattened Oblate Spheroid.—$m \ll 1$, $\mu = 1/m \gg 1$.

$$D_a = 1 + 2m^2 + \frac{8}{3}m^4 + \frac{16}{5}m^6 + \frac{128}{35}m^8 + \dots$$
$$-\frac{m\pi}{2}\left(1 + \frac{3}{2}m^2 + \frac{15}{8}m^4 + \frac{35}{16}m^6 + \frac{315}{128}m^8 + \dots\right). \quad (4.12)$$

$(\pi/2 = 1{\cdot}570\ 796.)$

Fig. 3.



Demagnetizing factors for oblate spheroids.

$m = a/b$; a, polar semi-axis; b, equatorial semi-axis. D_a, D_b, demagnetizing factors for polar, equatorial axis. The straight lines of slopes $-\pi/2$ and $+\pi/4$ are tangential to the curves.

This series up to the term in m^8 gives D_a to an accuracy of 1 unit in the seventh place at $m = 0{\cdot}1(\mu = 10)$, and to 1 unit in the fifth place at $m = 0{\cdot}2(\mu = 5)$. A graph of D_a and D_b against m is shown in fig. 3. The tangent to the D_a curve, of slope $-\pi/2$, and the tangent to the D_b curve, of slope $+\pi/4$, are also shown in the diagram.

The curves of fig. 3, it may be noted, are the same as the curves for $\beta = 1{\cdot}0$ in fig. 1.

Nearly Spherical Spheroid.—$1 - \mu^2 \ll 1$ (prolate) and $(1 - m^2) \ll 1$ (oblate). When the eccentricity, ϵ, approaches zero, series in ϵ^2, that is in $(1 - m^2)$ or $(1 - \mu^2)$, are more convenient than series in $(1 - m)$ or $(1 - \mu)$. These series are readily developed from (4.4) and (4.6).

For the prolate spheroid, with $a>b$, $\epsilon^2=1-(b/a)^2=1-\mu^2\ll 1$,

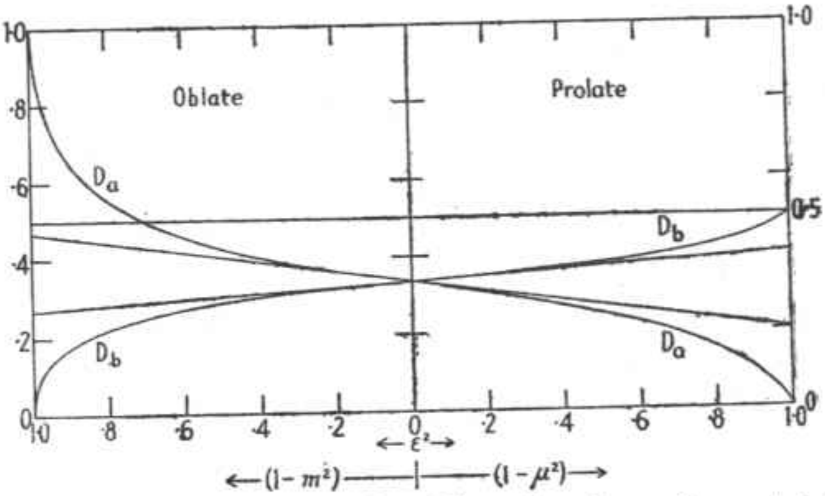
$$D_a=\frac{1}{3}-\frac{2\epsilon^2}{15}\left(1+\frac{3}{7}\epsilon^2+\frac{5}{21}\epsilon^4+\ldots\right). \quad \ldots \quad (4.13)$$

For the oblate spheroid, with $a<b$, $\epsilon^2=1-(a/b)^2=1-m^2\ll 1$,

$$D_x=\frac{1}{3}+\frac{2\epsilon^2}{15}\left(1+\frac{4}{7}\epsilon^2+\frac{8}{21}\epsilon^4+\ldots\right). \quad \ldots \quad (4.14)$$

These series, though appropriate for small eccentricities, are of limited utility with reference to the m or μ range; with (4.13), which is less slowly convergent than (4.14), it is necessary to carry the series to the term in ϵ^{10} to obtain an accuracy of 1 unit in the sixth place even for $m=1{\cdot}1$. The series are of interest, however, in giving at once for the sphere ($m=\mu=1$) the value for D_a of 1/3, which is not an immediately obvious limit to the expressions (4.2) to (4.6).

Fig. 4.



Demagnetizing factors for spheroids as functions of eccentricity.

$m=a/b$; $\mu=b/a$; a, polar semi-axis; b, equatorial semi-axis; ϵ, eccentricity. For prolate spheroid, $\epsilon^2=1-\mu^2$; for oblate spheroids, $\epsilon^2=1-m^2$. The straight lines of slopes $-2/15$ and $+1/15$ are tangential to the curves at $\epsilon^2=0$.

A graphical representation of the dependence of the demagnetizing factors on the eccentricity is given in fig. 4, the straight lines being the tangents to the curves at $\epsilon^2=0$, as given by (4.13) and (4.14); the slopes of the tangents are $-2/15$ for the D_a curve, and $+1/15$ for the D_b curve.

A consideration of the series (4.11) to (4.14) shows that their practical usefulness is limited to very restricted ranges of the values of the dimensional ratio, $m=a/b$, of an ellipsoid of revolution. Numerical evaluation of the expressions (4.2) and (4.5) is in general necessary, and even in the ranges to which the series are in practice applicable, the series expressions are comparable in convenience only for large and small values of m (say $m>10$ and $m<0{\cdot}1$) and for values in the neighbourhood of unity (say $m=1\pm 0{\cdot}05$). The series expressions do.

Table I.

Demagnetizing Factors for Ellipsoids of Revolution.

m-Table.

$m=a/b$; a, polar semi-axis ; b, equatorial semi-axis. $m<1$, oblate spheroid ; $m>1$, prolate spheroid. The demagnetizing field is given by $H=-4\pi DI$. $D_b=\frac{1}{2}(1-D_a)$. ($4\pi=12{\cdot}566\ 371$.)

m.	D_a.	*m*.	D_a.	*m*.	D_a.
0·0	1·000 000	3·5	0·089 651	20	0·006 749
0·1	0·860 804	3·6	86 477	21	6 230
0·2	750 484	3·7	83 478	22	5 771
0·3	661 350	3·8	80 641	23	5 363
0·4	588 154	3·9	77 954	24	4 998
0·5	0·527 200	4·0	0·075 407	25	0·004 671
0·6	475 826	4·1	72 990	30	3 444
0·7	432 065	4·2	70 693	35	2 655
0·8	394 440	4·3	68 509	40	2 116
0·9	361 822	4·4	66 431	45	1 730
1·0	0·333 333	4·5	0·064 450	50	0·001 443
1·1	308 285	4·6	62 562	60	1 053
1·2	286 128	4·7	60 760	70	0 805
1·3	266 420	4·8	59 039	80	0 637
1·4	248 803	4·9	57 394	90	0 518
1·5	0·232 981	5·0	0·055 821	100	0·000 430
1·6	218 713	5·5	48 890	110	363
1·7	205 794	6·0	43 230	120	311
1·8	194 056	6·5	38 541	130	270
1·9	183 353	7·0	34 609	140	236
2·0	0·173 564	7·5	0·031 275	150	0·000 209
2·1	164 585	8·0	28 421	200	125
2·2	156 326	8·5	25 958	250	083
2·3	148 710	9·0	23 816	300	060
2·4	141 669	9·5	21 939	350	045
2·5	0·135 146	10	0·020 286	400	0·000 036
2·6	129 090	11	17 515	500	24
2·7	123 455	12	15 297	600	17
2·8	118 203	13	13 490	700	13
2·9	113 298	14	11 997	800	10
3·0	0·108 709	15	0·010 749	900	0·000 008
3·1	104 410	16	09 692	1000	7
3·2	100 376	17	08 790	1100	6
3·3	096 584	18	08 013	1200	5
3·4	093 015	19	07 339	1300	4

TABLE II.

Demagnetizing Factors for Ellipsoids of Revolution.

μ-Table.

$\mu=b/a=1/m$; a, polar semi-axis; b, equatorial semi-axis. $\mu<1$, prolate spheroid; $\mu>1$, oblate spheroid. The demagnetizing field is given by $H=-4\pi DI$. $D_b=\frac{1}{2}(1-D_a)$. $(4\pi=12{\cdot}566\ 371.)$

μ.	D_a.	μ.	D_a.	μ.	D_a.
0·0	0·000 000	3·5	0·673 006	20	0·926 181
0·1	020 286	3·6	679 625	21	929 494
0·2	055 821	3·7	685 984	22	932 522
0·3	095 370	3·8	692 097	23	935 301
0·4	135 146	3·9	697 979	24	937 860
0·5	0·173 564	4·0	0·703 641	25	0·940 224
0·6	209 962	4·1	709 097	30	949 778
0·7	244 110	4·2	714 357	35	956 700
0·8	275 992	4·3	719 432	40	961 944
0·9	305 689	4·4	724 330	45	966 056
1·0	0·333 333	4·5	0·729 061	50	0·969 366
1·1	359 073	4·6	733 633	60	974 359
1·2	383 059	4·7	738 055	70	977 961
1·3	405 437	4·8	742 332	80	980 673
1·4	426 344	4·9	746 473	90	982 790
1·5	0·445 906	5·0	0·750 484	100	0·984 490
1·6	464 237	5·5	768 780	110	985 885
1·7	481 442	6·0	784 585	120	987 048
1·8	497 615	6·5	798 373	130	988 034
1·9	512 843	7·0	810 506	140	988 881
2·0	0·527 200	7·5	0·821 265	150	0·989 616
2·1	540 758	8·0	830 870	200	992 196
2·2	553 578	8·5	839 497	250	993 749
2·3	565 717	9·0	847 288	300	994 786
2·4	577 227	9·5	854 359	350	995 528
2·5	0·588 154	10	0·860 804	400	0·996 085
2·6	598 539	11	872 125	500	996 866
2·7	608 422	12	881 743	600	997 388
2·8	617 837	13	890 017	700	997 760
2·9	626 817	14	897 210	800	998 040
3·0	0·635 389	15	0·903 520	900	0·998 257
3·1	643 581	16	909 101	1000	998 431
3·2	651 417	17	914 071	1100	998 574
3·3	658 920	18	918 526	1200	998 692
3·4	666 110	19	922 542	1300	998 793

however, bring out the character of the variation, and they have value in enabling a rough estimate of the demagnetizing factors to be easily obtained.

5. *Numerical Tables.*

Numerical values of the demagnetizing factors for an ellipsoid of revolution, polar semi-axis a, equatorial semi-axis b, have been calculated from the expressions (4.2) and (4.5) which are repeated here for convenience :

$$D_a=\frac{1}{m^2-1}\left[\frac{m}{(m^2-1)^{1/2}}\ln\{m+(m^2-1)^{1/2}\}-1\right] \quad \text{for } a/b=m>1. \quad (5.1)$$

$$D_a=\frac{1}{1-m^2}\left[1-\frac{m}{(1-m^2)^{1/2}}\cos^{-1}m\right] \quad \text{for } a/b=m<1. \quad . \quad . \quad . \quad (5.2)$$

The demagnetizing factor along the equatorial axis is given by

$$D_b=\frac{1}{2}(1-D_a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.3)$$

An accuracy higher than 1 in 10^3 or at most 1 in 10^4 will seldom be required in physical applications. For the smaller values of D_a, however, a considerably lower accuracy will be sufficient, for it is the absolute rather than the relative accuracy which is relevant. The demagnetizing field is given by

$$H=-4\pi DI, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.4)$$

and the highest values of $4\pi I$ are of the order 10^4. (For iron, for example, the saturation value of $4\pi I$ is about $2{\cdot}16\times10^4$, for 34 per cent. ferrocobalt about $2{\cdot}37\times10^4$.) Thus, even in extreme cases, a value of D_a correct to 1 in the fifth place will give H to about 0·1 oersted. Six place values of D_a will therefore adequately cover all likely requirements. To ensure accuracy to the sixth place the calculations have been carried out so as to give an accuracy to 1 unit in the seventh place, and the rounded six place values are presented in the tables.

It is believed that the most useful form of the tables is with the dimensional ratio as argument. With $m(=a/b)$ as argument, a number of changes of interval are necessary if the relevant range is to be covered with a reasonable number of entries. The smallest interval used is 0·1 over the range $0\leqslant m\leqslant 5$. This interval, however, is too large over the range $0\leqslant m\leqslant 1$, over which D_a changes from 1 to 1/3, but the character of the variation is such that the range can be covered more conveniently by a complementary table with $\mu(=1/m)$ as argument than by the use of intervals smaller than 0·1 in the m table. Two tables, complementary to each other, have therefore been drawn up, an m table (Table I.) and a μ table (Table II.), each with 105 entries, the numerical values of the arguments being the same in each. In the μ table there are 95 entries for $\mu\geqslant 1$, covering the same range as is covered by the 11 entries for $0\leqslant m\leqslant 1$ in the m table; and conversely. In general the

m table will be appropriate for prolate spheroids ($m>1$, $\mu<1$) and the μ table for oblate spheroids ($m<1$, $\mu>1$).

In the calculations, use was made of Barlow's Tables (Comrie, 1941) for the powers and roots, and for the logarithms of Chambers's seven-place tables (ed. Pryde, 1937), supplemented in certain ranges by the Peters's ten-place tables (1922). No suitable table of the inverse circular functions being available, the inverse cosines were obtained by inverse interpolation in Gifford's eight-place table of natural sines (1914). A Brunsviga 20×12 calculating machine was used.

In each range of the two tables the numbers, usually seven- or eight-place, were checked by the method of differences. Additional checks were provided by cross interpolation between the two tables, and by the use of the series expansions (4.11) to (4.14).

Interpolation may be carried out using Bessel's formula. The necessary coefficients are given by Comrie (1936). The m table would ordinarily be used for $m>1$ and the μ table for $\mu>1$, but for values slightly less or greater than 1 the μ table will be found more convenient. For the greater part of each of the ranges in the tables, it is unnecessary to take into account differences beyond the second to obtain the tabular accuracy in the interpolates, but in the first part of each range the contribution from the fourth difference is not negligible. For the accuracy which is likely to be required in applications differences beyond the second will seldom be required.

It is a pleasure to acknowledge my indebtedness to Mr. C. W. Gilham, of the Department of Mathematics, University of Leeds, for guidance on the reduction of elliptic integrals.

Note added in proof (28 May, 1946).

Since this paper was submitted a paper has appeared, by J. A. Osborn (Phys. Rev. lxvii. p. 351, 1945), on the "Demagnetizing Factors of the General Ellipsoid." Expressions are given for the demagnetizing factors in forms essentially the same as those obtained in section 3 of the present paper, and there are three figures together equivalent to fig. 1, but on a larger scale, and with the curves at smaller intervals. In addition two numerical tables are given for ellipsoids with three unequal axes. The first of these gives the values of c/a, b/a and the associated demagnetizing factors corresponding to appropriate sets of equidistant tabular values in the elliptic integral tables. The second gives values of the demagnetizing factors for b/a at intervals of 0·1 from 0·1 to 1·0, in each case for a series of values of c/a, this set of values having been obtained by graphical interpolation from the first set. The values, it is stated, "are accurate to three decimal places and are probably in error several units in the fourth place." This accuracy seems adequate for any likely requirements in connection with the general ellipsoid, and the tables form a useful complement to those given here for the ellipsoid of revolution.

Summary.

The paper deals with the demagnetizing factors, D, of ellipsoids, D, being defined by the relation $H = -4\pi DI$, where H is the demagnetizing field and I the intensity of magnetization, H and I being co-directional.

The derivation of the usual integral formulae for the demagnetizing factors for the three principal axes of an ellipsoid with three unequal axes is outlined. These integral formulae are re-expressed in terms of normal elliptic integrals of the first and second kinds, enabling the factors to be evaluated without difficulty with the aid of standard tables of these integrals. The variation of the three demagnetizing factors with the ratio of the smallest to the greatest principal axis for a series of constant values of the ratio of the intermediate to the greatest axis is shown in a figure. Particular curves in this figure correspond to an ellipsoid of revolution.

The formulae for the demagnetizing factors, D_a and D_b of an ellipsoid of revolution (a, polar semi-axis; b, equatorial semi-axis) are derived, and the various alternative forms, and the relations between them and the expressions for an ellipsoid with three unequal axes are discussed. Series expansions are obtained in terms of the dimensional ratio, $m(m=a/b)$, or its inverse, $\mu(\mu=b/a)$, suitable for $m \gg 1$, $m \ll 1$ and $m \doteq 1$. Values of D_a are given to six places of decimals in two numerical tables, an m-table and a μ-table, each with 105 entries. Values of the argument, m or μ, range from 0 to 1300, the range being covered with the intervals shown in brackets: 0·0 (0·1) 5·0 (0·5) 10·0 (1) 25 (5) 50 (10) 150 (50) 400 (100) 1,300. The first table is most useful for prolate spheroids ($m>1$), the second for oblate spheroids ($\mu=1/m>1$). The demagnetizing factor along an equatorial axis, D_b, is not tabulated, as it is easily obtained from D_a, being equal to $\frac{1}{2}(1-D_a)$.

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XCVIII. *The Technique of Precise Electron-Diffraction Measurements with Polycrystalline Specimens.*

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[Plate XI.]

Abstract.

Some factors affecting the size and shape of the electron-diffraction rings obtained with transmission specimens are considered. The most important of these are the accumulation of electric charge on the photographic plate and the presence of weak stray magnetic fields.

In an earlier paper [(3)] we have discussed the problem of the measurement of the radii of the Debye–Scherrer rings produced by the diffraction of electrons by polycrystalline specimens, and we have shown that it is possible to measure these radii with an accuracy of about one part in ten thousand. Measurements of this precision reveal a number of unexpected features, which are partly instrumental in origin and in part are related to the crystal structure of the specimen. These peculiarities have mostly escaped notice because the accuracy of previous

* Communicated by Prof. J. A. Crowther, M.A., Sc.D.